

SUPERSYMMETRIC KALUZA–KLEIN REDUCTIONS OF M-WAVES AND MKK-MONOPOLES

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ABSTRACT. We investigate the Kaluza–Klein reductions to ten dimensions of the purely gravitational half-BPS M-theory backgrounds: the M-wave and the Kaluza–Klein monopole. We determine the moduli space of smooth (supersymmetric) Kaluza–Klein reductions by classifying the freely-acting space-like Killing vectors which preserve some Killing spinor. As a consequence we find a wealth of new supersymmetric IIA configurations involving composite and/or bound-state configurations of waves, D0 and D6-branes, Kaluza–Klein monopoles in type IIA and flux/nullbranes, and some other new configurations. Some new features raised by the geometry of the Taub–NUT space are discussed, namely the existence of reductions with no continuous moduli. We also propose an interpretation of the flux 5-brane in terms of the local description (close to the branes) of a bound state of D6-branes and ten-dimensional Kaluza–Klein monopoles.

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1. INTRODUCTION AND CONCLUSIONS

In this paper we investigate and classify the possible supersymmetric Kaluza–Klein reductions of the purely gravitational half-BPS M-theory backgrounds to ten dimensions. It is thus an extension of the ideas and techniques developed in [?] and [?] and applied to eleven-dimensional Minkowski spacetime and the elementary half-BPS M2- and M5-branes, respectively.

A purely gravitational supersymmetric M-theory background (M, g) consists of an eleven-dimensional lorentzian spin Ricci-flat manifold admitting parallel spinors. We remark, as discussed for example in [?], that in lorentzian geometry the existence of parallel spinors does not imply Ricci-flatness. The existence of parallel spinors does however constraint the holonomy to belong to the subgroup $\text{Spin}(7) \ltimes \mathbb{R}^9$ of the eleven-dimensional Lorentz group [?, ?]. There are precisely two half-BPS possibilities: either a gravitational wave [?] (which can be delocalised along one or more transverse directions) or the product $\mathbb{R}^{1,6} \times N$ where N is a four-dimensional hyperkähler manifold. For N the Taub–NUT gravitational instanton, this is the eleven-dimensional Kaluza–Klein monopole [?, ?, ?].

The standard Kaluza–Klein reductions of these spaces give rise to well-known IIA backgrounds: wave, D0-brane, Kaluza–Klein monopole and D6-brane and, just as for flat space and the M2- and M5-branes, considering more general (twisted) reductions one obtains backgrounds where fluxbranes and nullbranes have been added. Indeed, one of our main aims is to give a complete list of the supersymmetric composite configurations made of waves, D0-branes, D6-branes, Kaluza–Klein monopoles in type IIA with fluxbranes and nullbranes.

The techniques and methodology used in this paper are fully explained in [?], to which we refer the reader for further details, particularly the introductory section, as well as for a more complete list of references. Let us simply restate very briefly the main idea. Given a purely gravitational M-theory background (M, g) with isometry group G , we would like to determine in the first instance all one-parameter subgroups $\Gamma \subset G$ whose orbits in M are spacelike and such that the quotient M/Γ is smooth. We then wish to single out those subgroups for which the resulting IIA background $(M/\Gamma, h, \Phi, A_1)$ is supersymmetric. Each one-parameter subgroup Γ is generated by a Killing vector ξ , which we identify with an element of the Lie algebra \mathfrak{g} of G . We are free to conjugate by G , since conjugate elements in \mathfrak{g} give rise to isometric reductions, and we are also free to rescale the Killing vector, since this corresponds to a reparametrisation of the orbit. Thus we are interested in the following equivalence relation in \mathfrak{g} :

$$X \sim tgXg^{-1} \quad \text{where } X \in \mathfrak{g}, t \in \mathbb{R}^\times \text{ and } g \in G.$$

The quotient of \mathfrak{g} by this equivalence relation defines the moduli space of one-parameter subgroups of G . Selecting from this moduli space those subgroups for which the orbits are spacelike and the quotient along the orbits is smooth, we arrive at the *moduli space of smooth reductions*. Within this space there are loci corresponding to those reductions which are also supersymmetric. These loci comprise the *moduli space of supersymmetric Kaluza–Klein reductions*, and one of the main results in this paper is the determination of this space for the M-wave and the Kaluza–Klein monopole. These conditions on the reduction translate into conditions on the Killing vector used to reduce, which we fully analyse.

As mentioned in [?], it is possible to stop short of the reduction to IIA and consider new M-theory backgrounds obtained by quotienting by a (discrete) cyclic subgroup $\Gamma_0 \subset \Gamma$. In the case where Γ is noncompact (i.e., diffeomorphic to \mathbb{R}) one has that Γ_0 is infinite cyclic and hence isomorphic to \mathbb{Z} , whereas for Γ a circle subgroup, which will occur in the Kaluza–Klein monopole, Γ_0 will be isomorphic to \mathbb{Z}_N for some N . Quotienting by Γ_0 thus gives rise to M-theory backgrounds which are locally isometric to the original background. Although we do not emphasise these constructions in this paper, let us point out that from our results there also follows more or less straightforwardly a classification of such “orbifolds” where the group Γ_0 has one generator in the image of the exponential map.

Having determined the supersymmetric Kaluza–Klein reductions, we then use the techniques explained in detail in [?] to pass to adapted coordinates and write

down the corresponding IIA background explicitly in order to further identify in terms of composites or bound states of D-branes, Kaluza–Klein monopoles, waves, fluxbranes or nullbranes. Briefly, we change coordinates from, say, (z, \mathbf{y}) , where $\xi = \partial_z + \alpha$, α standing for an arbitrary element of \mathfrak{g} commuting with ∂_z , to an adapted coordinate system (z, \mathbf{x}) defined by

$$\mathbf{x} = U\mathbf{y} \quad , \quad U = \exp(-z\alpha) \quad \text{such that} \quad \xi(z, \mathbf{x}) = \partial_z \quad . \quad (1.1)$$

Since α acts affinely, the reduction manifestly depends on a constant matrix B and a constant vector \mathbf{C} defined by

$$\alpha\mathbf{y} = B\mathbf{y} + \mathbf{C} \quad . \quad (1.2)$$

It is then straightforward to perform the actual reductions, as explained in detail in [?].

Even though it is not emphasised in this work, it should be clear that by applying the usual dualities, one could construct a wealth of new supersymmetric configurations involving duals of fluxbranes and nullbranes [?] and duals of standard waves, D0-branes, Kaluza–Klein monopoles and D6-branes.

Let us now summarise the main results of this paper. As already mentioned we classify the supersymmetric configurations in type IIA supergravity involving waves, Kaluza–Klein monopoles, D0-branes, D6-branes, fluxbranes and nullbranes. These results are summarised in Tables 2, 4, 5, 8, 9 and 10. As already stressed in [?] for the M2 and M5-branes, we find new backgrounds, not only associated with bound states of waves and D0-branes or monopoles and D6-branes in the flux/nullbrane sectors of the theory, but also in the case of the delocalised M-wave, other backgrounds with a more elusive interpretation, obtained by reducing along the orbits of Killing vectors which involve time/lightlike translations and transverse rotations.

The analysis of the supersymmetric reductions of the Kaluza–Klein monopole reveals some interesting features. In the first place there exist reductions with only discrete moduli. Due to the fact that the Kaluza–Klein monopole has freely-acting Killing vectors with compact orbits, there are further requirements to the ones discussed in [?] to be satisfied to get a smooth spin manifold. In particular, the integral curves of these Killing vectors need to be periodic and this fact manifests itself in the integrality of the parameters defining the Killing vector. Supersymmetry then imposes further linear Diophantine equations on these parameters, resulting in discrete regimes in moduli space. Second, there exist fluxbranes constructed out of the 3-spheres that foliate the Taub–NUT geometry. The action of Killing vectors on the Killing spinors of the Taub–NUT space (see Section B.2) reveals the possibility of constructing supersymmetric fluxbranes by performing Kaluza–Klein reductions along the orbits of a Killing vector involving not only a spacelike translation along the monopole, but also both a rotation on the monopole and an element of $SU(2)$ acting naturally on the 3-spheres foliating the Taub–NUT geometry. Finally, we give a novel interpretation of the flux 5-brane [?] as the local description (close to the branes) of a bound state of IIA Kaluza–Klein monopoles and D6-branes[?]. This will be argued in terms of the supersymmetry preserved by both systems, and furthermore, by explicitly studying the supergravity configuration describing the bound state close to the branes.

The paper is organised as follows. In Section 2, we apply our technology both to the M-wave and its delocalisation along one transverse direction. In Section 3 the same is done for the Kaluza–Klein monopole. Some technical points concerning group theory and spinors, Killing spinors of the Taub–NUT geometry and the action of the isometry group on them are left to the corresponding appendices. Finally,

an appendix is included where we analyse the question of the existence of spin structures on quotients of the Kaluza–Klein monopole.

2. KALUZA–KLEIN REDUCTIONS OF THE M-WAVE

In this section we classify the set of IIA backgrounds obtained by reducing the M-wave along the orbits of a one-parameter subgroup of the isometry group. In Section 2.1 we discuss the M-wave and in Section 2.2 we discuss the M-wave which has been delocalised along one transverse direction.

2.1. M-wave. In this section we discuss the supersymmetric Kaluza–Klein reductions of the purely gravitational M-wave [?]

$$g = 2dy^+dy^- + 2V(dy^+)^2 + ds^2(\mathbb{E}^9) , \quad (2.1)$$

where V is a harmonic function on \mathbb{E}^9 . The maximally symmetric solution corresponds to a function V which only depends on the transverse radius r given by $r^2 = \sum_i y^i y^i$. Demanding that the spacetime be asymptotically flat at large r means that $\lim_{r \rightarrow \infty} V(r)$ should be a constant. This constant can be reabsorbed by a change of variables, whence a convenient choice of V which makes the metric manifestly asymptotically flat is

$$V(r) = \frac{Q}{r^7} , \quad \text{for some } Q > 0.$$

In the absence of F_4 , the Killing spinors are parallel relative to the spin connection. In the above coordinates, such spinors are constant

$$\varepsilon = \varepsilon_\infty \quad (2.2)$$

and obey

$$\Gamma_+ \varepsilon_\infty = 0 . \quad (2.3)$$

The isometry group is

$$G = \text{SO}(9) \times \mathbb{R}^2 \subset \text{ISO}(1, 10) , \quad (2.4)$$

where \mathbb{R}^2 corresponds to translations along the lightcone directions y^\pm , and the $\text{SO}(9)$ is the transverse rotation group. The Lie algebra is given by

$$\mathfrak{g} = \mathfrak{so}(9) \times \mathbb{R}^2 , \quad (2.5)$$

whence any Killing vector can be decomposed as

$$\xi = \tau + \rho , \quad (2.6)$$

with $\tau = a\partial_+ + b\partial_-$ and $\rho \in \mathfrak{so}(9)$.

2.1.1. Freely-acting spacelike isometries. By conjugating with G , we may bring ξ to a normal form. In practice this means conjugating ρ to belong to a fixed Cartan subalgebra. For example, we can choose

$$\rho = \theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + \theta_4 R_{78} , \quad (2.7)$$

where R_{ij} stands for the generator of a rotation in the (ij) -plane. The norm of a Killing vector $\xi = a\partial_+ + b\partial_- + \rho$ is given by

$$\|\xi\|^2 = 2ab + 2Va^2 + \|\rho\|^2 ,$$

which is bounded below by the flat norm of the translation component:

$$\|\xi\|^2 \geq 2ab = \|\tau\|_\infty^2$$

That this bound is sharp can be shown by simply looking at large r along any direction fixed by ρ : y^9 in the above example. Therefore ξ is spacelike if and only if τ is spacelike relative to the flat metric. It is more convenient to change coordinates

from lightcone (y^\pm) to pseudo-euclidean ($y^0, y^{\mathfrak{d}}$) such that $\tau = c\partial_0 + d\partial_{\mathfrak{d}}$. The condition that τ be asymptotically spacelike says that $d^2 > c^2$, whence d cannot be zero. Using the freedom to rescale we can set $d = 1$, leaving a one parameter family of translations $\tau = \partial_{\mathfrak{d}} + c\partial_0$ with $-1 < c < 1$. The action generated by such a Killing vector is always free, and hence the moduli space of smooth reductions of the M-wave is five-dimensional and is parametrised by the cartesian product of the interval $(-1, 1)$ with a fixed Cartan subalgebra of $\mathfrak{so}(9)$. As we will see below, supersymmetry will select a locus with codimension one.

2.1.2. Absence of closed causal curves. We would like to prove that the M-wave background (2.1) reduced along the orbits of $\xi = \partial_{\mathfrak{d}} + a\partial_0$ ($|a| < 1$) has no closed causal curves. As already stressed in the previous subsection, the constraint $|a| < 1$ comes from demanding that ξ be everywhere spacelike. The norm of ξ is given by

$$\|\xi\|^2 = (1-a)[(1+a) + (1-a)V] .$$

In adapted coordinates to the action of ξ , the background (2.1) takes the form

$$g = (V-1)(dt)^2 + \|\xi\|^2(dz)^2 - 2(1-a)Vdz \cdot dt + ds^2(\mathbb{E}^9) .$$

Suppose, for a contradiction, that a causal curve $x(\lambda)$ does exist joining the points (t_0, z_0, x_0^i) and $(t_0, z_0 + \Delta, x_0^i)$ for $i = 1, 2, \dots, 9$. It follows that there must exist at least one value λ^* of the affine parameter where the timelike component of the tangent vector to the causal curve must vanish. Computing the norm of the tangent vector at that point, one derives the inequality

$$\|\xi\|^2(\lambda^*) \left. \frac{dz}{d\lambda} \right|_{\lambda^*}^2 + \sum_i \left. \frac{dx^i}{d\lambda} \right|_{\lambda^*}^2 \leq 0 ,$$

which can never be satisfied unless the tangent vector to the causal curve vanishes identically at λ^* , violating the hypothesis that λ is an affine parameter. Therefore we conclude there are no such closed causal curves.

2.1.3. Supersymmetry. As usual the translation component of ξ is not constrained by supersymmetry, but the rotation component is constrained to lie in the isotropy of a parallel spinor obeying (2.3). In [?] we explained how to determine the supersymmetric locus in the parameter space and we refer to that paper for details. Let S_{11} denote the irreducible spinor representation of $\text{Spin}(1, 10)$ and let $S_{11}^+ = \ker \Gamma_+ \subset S_{11}$ denote the space of Killing spinors of the M-wave. Under the transverse spin group $\text{Spin}(9)$, S_{11}^+ is isomorphic to the unique irreducible spinor representation S_9 . As discussed in Appendix A, relative to the basis dual to the R_{ij} in (2.7), the subspace S_{11}^+ has weights $(\pm 1, \pm 1, \pm 1, \pm 1)$ where the signs are uncorrelated, for a total of 16 weights. Therefore ρ will annihilate a Killing spinor if and only if the θ_i belong to the union of the eight hyperplanes

$$\sum_{i=1}^4 \mu_i \theta_i = 0 , \quad \text{with } \mu_i^2 = 1. \quad (2.8)$$

(As usual there are only eight hyperplanes because (μ_i) and $(-\mu_i)$ define the same hyperplane.) A generic ρ in one of these hyperplanes will annihilate a two-dimensional subspace of Killing spinors and hence the associated reduction will preserve $\frac{1}{8}$ of the supersymmetry preserved by the M-wave, or a fraction $\nu = \frac{1}{16}$ of the supersymmetry of the eleven-dimensional vacuum. This corresponds to ρ belonging to an $\mathfrak{su}(4)$ subalgebra. This is clearly a four-dimensional locus of the moduli space of smooth reductions.

There is enhancement of supersymmetry if the rotation ρ belongs to two or more hyperplanes. There are two kinds of pairwise intersections: those planes where none of the θ_i vanish but two pairs do, say $\theta_1 + \theta_2 = 0$ and $\theta_3 + \theta_4 = 0$. Such

rotations belong to an $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$ subalgebra and preserve a fraction $\nu = \frac{1}{8}$ of the supersymmetry. The other kind of pairwise intersection is when only one of the θ_i vanishes, say $\theta_1 = 0$ and hence $\theta_2 + \theta_3 + \theta_4 = 0$. Such a rotation belongs to an $\mathfrak{su}(3)$ subalgebra and the reduction preserves again $\frac{1}{8}$ of the supersymmetry. These loci are three-dimensional. If a rotation belongs to three of these hyperplanes, then two of the θ_i must vanish. This means that ρ belongs to an $\mathfrak{su}(2)$ subalgebra and the reduction preserves $\frac{1}{4}$ of the supersymmetry. This locus is two-dimensional. Finally there is one point in the intersection of all hyperplanes, corresponding to $\rho = 0$. This gives rise to a one-dimensional locus of $\frac{1}{2}$ -BPS reductions. These observations are summarised in Table 1.

Translation	Subalgebra	ν	dim
$\partial_{\mathfrak{t}} + a\partial_0$	$\mathfrak{su}(4)$	$\frac{1}{16}$	4
	$\mathfrak{su}(3)$	$\frac{1}{8}$	3
	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$	$\frac{1}{8}$	3
$-1 < a < 1$	$\mathfrak{su}(2)$	$\frac{1}{4}$	2
	$\{0\}$	$\frac{1}{2}$	1

TABLE 1. Supersymmetric reductions of the M-wave. We indicate the spinor isotropy subalgebra to which the rotation belongs, the fraction ν of the supersymmetry preserved and the dimension of the corresponding stratum of the moduli space \mathcal{M} of supersymmetric reductions.

We would like to stress that by restricting to (discrete) cyclic subgroups $\Gamma_0 \subset \Gamma$ generated by $\xi = \partial_{\mathfrak{t}} + \rho$, the corresponding quotient manifolds $\mathcal{M}_{\text{wave}}/\Gamma_0$ would be describing the propagation of M-waves in an eleven-dimensional fluxbrane.

2.1.4. Explicit reductions. There is a single type of reduction to be analysed for the M-wave, the one whose Killing vector is written as $\xi = \partial_{\mathfrak{t}} + \alpha$ where α stands for an infinitesimal affine transformation consisting of a rotation in the space transverse to the direction of propagation $z = x^{\mathfrak{z}}$ and a timelike translation

$$\begin{aligned} \alpha = & a\partial_0 + \theta_1(y^1\partial_2 - y^2\partial_1) + \theta_2(y^3\partial_4 - y^4\partial_3) \\ & + \theta_3(y^5\partial_6 - y^6\partial_5) + \theta_4(y^7\partial_8 - y^8\partial_7), \end{aligned}$$

the parameter a being bounded above in absolute value by $|a| < 1$.

The matrix B defined in (1.2) characterising the Kaluza–Klein reduction can be written in the basis $\{x^0, x^1, \dots, x^8\}$ of the adapted coordinate system (1.1) as

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_4 & 0 \end{pmatrix}. \quad (2.9)$$

Notice that x^9 is left invariant by the construction giving rise to these configurations. Since there is an extra translation operator besides $\partial_{\mathfrak{t}}$, there is a 9-vector

\mathbf{C} taking care of the inhomogeneous part in the change of coordinates (1.1). The transpose of this vector is given by

$$(\mathbf{C})^t = (a, \vec{0}) .$$

After the Kaluza–Klein reduction, one obtains the ten-dimensional metric

$$g = \Lambda^{1/2} \{ (V-1) (dx^0)^2 + ds^2(\mathbb{E}^9) \} - \Lambda^{3/2} A_1^2$$

where A_1 stands for the RR 1-form, which takes the form

$$A_1 = \Lambda^{-1} \left\{ dx^0 [V(1+a) - a] + (B \cdot x)^i \cdot dx_i \right\}$$

in terms of a scalar function Λ given by

$$\Lambda = (1+a) [(1-a) + V(1+a)] + (B \cdot x)^i (B \cdot x)_i . \quad (2.10)$$

The dilaton is also given in terms of Λ by $\Phi = \frac{3}{4} \log \Lambda$.

Let us, first of all, discuss the physical interpretation for the configurations described in the subspace $a = 0$. It should be clear at this stage, that these configurations describe composite configurations of D0-branes and fluxbranes. The absence of null rotations is telling us that there are no D0-branes in the nullbrane sector of string theory. For arbitrary values of the deformation parameters $\{\theta_i\}$, the configuration would break supersymmetry completely, and its interpretation would be in terms of composite configurations involving D0-branes at $r = 0$ and, generically, four different F7-branes lying at $x^1 = x^2 = 0$, $x^3 = x^4 = 0$, $x^5 = x^6 = 0$ and $x^7 = x^8 = 0$, respectively. It is the presence of the F7-branes that breaks supersymmetry completely.

ν	Object	Subalgebra
$\frac{1}{4}$	D0+F5	$\mathfrak{su}(2)$
$\frac{1}{8}$	D0+F3	$\mathfrak{su}(3)$
$\frac{1}{8}$	D0+F1	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$
$\frac{1}{16}$	D0+F1	$\mathfrak{su}(4)$

TABLE 2. Supersymmetric configurations of D0-branes and fluxbranes

On the other hand, there are four different types of supersymmetric configurations which are summarised in Table 2. The discussion of the taxonomy is entirely analogous to the one given for M2-branes in [?] when restricting to the above subclass of Killing vectors. We refer the reader to the corresponding subsection for further details. Notice, though, that in this case the D0-branes are not constrained to move on the fluxbranes, but everywhere on \mathbb{E}^9 , as already emphasized in [?].

Let us finally move to the subspace where $a \neq 0$. Let us first consider the background in which all θ_i parameters are set to zero. In that particular case, the ten dimensional configuration turns out to be

$$\begin{aligned} g &= -\Lambda^{-1/2} (dx^0)^2 + \Lambda^{1/2} ds^2(\mathbb{E}^9) \\ F_2 &= dA_1 = dx^0 \wedge d\Lambda^{-1} \\ \Phi &= \frac{3}{4} \log \Lambda , \end{aligned} \quad (2.11)$$

where the scalar function Λ in (2.10) reduces to

$$\Lambda = (1+a) [(1-a) + V(1+a)] .$$

Such a configuration has the same group of isometries as the standard solution describing D0-branes. Since $|a| < 1$, it is still asymptotically flat, but its RR charge

acquires an extra $(1+a)^2$ constant factor. Due to the fact that a is a continuous parameter, whenever $a \neq 0$, the charge will no longer be quantised. As expected, for $a = \pm 1$, the configuration becomes singular. The physical status for these configurations remains unclear to us. It is clear, though, that by switching the parameters θ_i in a supersymmetric preserving way, we are adding fluxbranes to the basic configuration (2.11).

2.2. Delocalised M-wave. In this section we discuss the supersymmetric Kaluza–Klein reductions of the purely gravitational M-wave when it has been delocalised in one transverse direction. This solution has metric

$$g = 2dy^+dy^- + 2V(dy^+)^2 + (dy)^2 + ds^2(\mathbb{E}^8) , \quad (2.12)$$

where V is a harmonic function on \mathbb{E}^8 and y stands for the transverse spacelike direction in which (2.1) was delocalised. The maximally symmetric solution corresponds to a function V which only depends on the transverse radius r given by $r^2 = \sum_i y^i y^i$. Again a convenient choice with the virtue of yielding a manifestly asymptotically flat spacetime is

$$V(r) = \frac{Q}{r^6} , \quad \text{for some } Q > 0.$$

The properties of its Killing spinors are exactly as in (2.2) and (2.3).

The isometry group is

$$G = \text{SO}(8) \times \mathbb{R}^3 \subset \text{ISO}(1, 10) , \quad (2.13)$$

where \mathbb{R}^3 corresponds to translations along the lightcone directions y^\pm and y , and the $\text{SO}(8)$ is the transverse rotation group. The Lie algebra is given by

$$\mathfrak{g} = \mathfrak{so}(8) \times \mathbb{R}^3 , \quad (2.14)$$

whence any Killing vector can be decomposed as

$$\xi = \tau + \rho , \quad (2.15)$$

with $\tau = a\partial_+ + b\partial_- + c\partial_y$ and $\rho \in \mathfrak{so}(8)$.

2.2.1. Freely-acting spacelike isometries. As usual we may bring ξ to a normal form by conjugating with G . Notice that ρ can always be chosen as in (2.7). In this way, the norm of the most general Killing vector is given by

$$\|\xi\|^2 = 2ab + 2Va^2 + c^2 + \|\rho\|^2 .$$

In contrast with the case of the M-wave treated above, the rotation now need not leave any direction invariant. This means that the norm of the rotation is bounded below by $r^2 m^2$, where m^2 is the minimum of the norm of ρ on the unit sphere in \mathbb{E}^8 , and as a result the norm of ξ is bounded below by

$$\|\xi\|^2 \geq \|\tau\|_\infty^2 + 2V(r)a^2 + r^2 m^2 .$$

If ρ is given by (2.7), then $m^2 = \min_i \theta_i^2$. This bound is sharp since there are points in the sphere for which $\|\rho\|^2 = r^2 m^2$. We must therefore distinguish between two cases:

- (a) $a = 0$. In this case $\|\xi\|^2 \geq c^2 + r^2 m^2$, whence it is bounded below by c^2 as r tends to 0. Therefore the Killing vector is everywhere spacelike if and only if $c \neq 0$.
- (b) $a \neq 0$. In this case the norm of ξ is bounded below by a function

$$f(r) = \|\tau\|_\infty^2 + \frac{2Qa^2}{r^6} + r^2 m^2 .$$

This function is bounded below. Ensuring that the lower bound is positive will impose a lower bound on $\|\tau\|_\infty^2$ allowing it to be negative, as was already

noticed for the M2-brane and the delocalised M5-brane in [?]. The function $f(r)$ grows without bound as $r \rightarrow 0$ and $r \rightarrow \infty$. There is a unique critical point $r_0 > 0$ given by

$$f'(r_0) = 0 \implies m^2 r_0^8 = 6Qa^2 .$$

Demanding that $f(r_0) > 0$ yields a lower bound on the asymptotic norm of the translation: $\|\tau\|_\infty^2 > -\mu^2$, where

$$\mu^2 = \frac{8}{6^{3/4}} Q^{1/4} |a|^{1/2} m^{3/2} .$$

In all cases the action is free provided that the translation component is present.

In summary, we can distinguish between different kinds of freely-acting spacelike Killing vectors:

- (A) $\xi = \partial_y + b\partial_- + \rho$, where we have already used the freedom to rescale and put $c = 1$, whence b and ρ are unconstrained. Such Killing vectors form a five-dimensional stratum of the moduli space of smooth reductions.
- (B) In this case $\xi = \partial_+ + b\partial_- + c\partial_y + \rho$, where we have used the freedom to rescale and the fact that $a \neq 0$ to set $a = 1$. We must distinguish between two cases:
 - (i) ρ does not fix any direction; whence $2b + c^2 > -\mu^2$ with

$$\mu^2 = \frac{8}{6^{3/4}} Q^{1/4} m^{3/2} . \quad (2.16)$$

Such Killing vectors give rise to a six-dimensional stratum of the moduli space of smooth reductions.

- (ii) ρ fixes some direction, so that one of the θ parameters in ρ vanishes. In this case, $2b + c^2 > 0$, so that the translation is spacelike relative to the flat norm. The corresponding stratum of the moduli space is five-dimensional.

In all cases, supersymmetry will select a codimension-one locus.

2.2.2. Supersymmetry. The analysis of supersymmetry is entirely analogous to the fully localised M-wave configuration, since all new possibilities discussed previously only involve translational isometries, which do not constrain supersymmetry. The supersymmetric reductions are summarised in Table 3.

2.2.3. Explicit reductions. Let us start our explicit reduction analysis by considering the reductions along the orbits of the Killing vector $\xi = \partial_y + \alpha$, where α stands for the infinitesimal affine transformation

$$\begin{aligned} \alpha = & b\partial_- + \theta_1(y^1\partial_2 - y^2\partial_1) + \theta_2(y^3\partial_4 - y^4\partial_3) \\ & + \theta_3(y^5\partial_6 - y^6\partial_5) + \theta_4(y^7\partial_8 - y^8\partial_7) . \end{aligned}$$

The constant matrix B defined in (1.2) is a 9×9 matrix which in the adapted coordinate system (1.1) does not act either on the x^+ or y directions. It is given explicitly, in the basis $\{x^-, x^1, \dots, x^8\}$, by

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_4 & 0 \end{pmatrix} . \quad (2.17)$$

Translation	Subalgebra	ν	dim
$\partial_y + b\partial_-$	$\mathfrak{su}(4)$	$\frac{1}{16}$	4
	$\mathfrak{su}(3)$	$\frac{1}{8}$	3
	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$	$\frac{1}{8}$	3
	$\mathfrak{su}(2)$	$\frac{1}{4}$	2
	$\{0\}$	$\frac{1}{2}$	1
$\partial_+ + b\partial_- + c\partial_y$ $2b + c^2 > -\mu^2$	$\mathfrak{su}(4)$	$\frac{1}{16}$	5
	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$	$\frac{1}{8}$	4
$\partial_+ + b\partial_- + c\partial_y$ $2b + c^2 > 0$	$\mathfrak{su}(3)$	$\frac{1}{8}$	4
	$\mathfrak{su}(2)$	$\frac{1}{4}$	3
	$\{0\}$	$\frac{1}{2}$	2

TABLE 3. Supersymmetric reductions of the delocalised M-wave. We indicate the form of the translation, the spinor isotropy subalgebra to which the rotation belongs, the fraction ν of the supersymmetry preserved and the dimension of the corresponding stratum of the moduli space \mathcal{M} of supersymmetric reductions. The parameter μ^2 is nonzero and is given in (2.16).

There is a nontrivial 9-vector \mathbf{C} taking care of the inhomogeneous part of the change of coordinates due to the existence of the extra translation ∂_- . In the same basis used above, it is given by

$$(\mathbf{C})^t = (a, \vec{0}) .$$

The ten dimensional configuration obtained by Kaluza–Klein reduction has a metric that takes the form

$$g = \Lambda^{1/2} \{ 2dx^+ dx^- + 2V(dx^-)^2 + ds^2(\mathbb{E}^8) \} - \Lambda^{3/2} A_1^2 ,$$

where A_1 is the RR 1-form potential, which together with the nontrivial dilaton profile are given by

$$\begin{aligned} A_1 &= \Lambda^{-1} (b dx^+ + (B \cdot x)^i dx_i) \\ \Phi &= \frac{3}{4} \log \Lambda \end{aligned}$$

in terms of the scalar function

$$\Lambda = 1 + (B \cdot x)^i (B \cdot x)_i .$$

If we restrict ourselves to the subspace defined by $b = 0$, and set all the rotation parameters θ_i to zero, one gets the standard wave solution in type IIA. It is then clear that by keeping $b = 0$ but turning on some of the θ_i , one would start generating new solutions in the fluxbrane sector. The classification and interpretation of the solutions is analogous to the ones found before and are summarised in Table 4.

On the other hand, we can study the family of solutions characterised by vanishing θ_i , but having a nonvanishing extra translation parameter ($b \neq 0$). Notice that the scalar function becomes trivial ($\Lambda = 1$) and the RR 1-form potential $A_1 = b dx^+$ becomes pure gauge. We are thus left with a purely gravitational configuration with

ν	Object	Subalgebra
$\frac{1}{4}$	WA+F5	$\mathfrak{su}(2)$
$\frac{1}{8}$	WA+F3	$\mathfrak{su}(3)$
$\frac{1}{8}$	WA+F1	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$
$\frac{1}{16}$	WA+F1	$\mathfrak{su}(4)$

TABLE 4. Supersymmetric configurations of type IIA waves (WA) and fluxbranes

a wave metric

$$g = 2dx^+ dx^- + (2V - b^2)(dx^+)^2 + ds^2(\mathbb{E}^8) .$$

Notice that this spacetime is again asymptotically flat in the limit $r \rightarrow \infty$, since in this limit $2V - b^2$ tends to a constant. Turning on the angle parameters θ_i one is just adding fluxbranes to the above configuration.

There is a second inequivalent set of Kaluza–Klein reductions that one can study for these backgrounds. These are the reductions along the orbits of the Killing vector $\xi = \partial_+ + \alpha$, where α stands for the generators of transverse rotations in \mathbb{E}^8 and translations in the $\{x^-, y\}$ directions :

$$\begin{aligned} \alpha = b\partial_- + c\partial_y + \theta_1(y^1\partial_2 - y^2\partial_1) + \theta_2(y^3\partial_4 - y^4\partial_3) \\ + \theta_3(y^5\partial_6 - y^6\partial_5) + \theta_4(y^7\partial_8 - y^8\partial_7) . \end{aligned}$$

The constant matrix B defined in (1.2) is a 10×10 matrix. It is given explicitly, in the adapted coordinate (1.1) basis $\{x^-, y, x^1, \dots, x^8\}$, by

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \theta_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\theta_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \theta_4 & 0 \end{pmatrix} . \quad (2.18)$$

There is a nontrivial 10-vector \mathbf{C} taking care of the inhomogeneous part of the change of coordinates (1.1) due to the existence of the extra translations ∂_+ and ∂_y . In the same basis used above, it is given by

$$(\mathbf{C})^t = (a, c, \vec{0}) .$$

After reduction, the ten-dimensional metric becomes

$$g = \Lambda^{1/2} \{ ds^2(\mathbb{E}^8) + (dx)^2 \} - \Lambda^{3/2} (A_1)^2 ,$$

where $dy = dx + cdx^+$ and A_1 is the RR 1-form potential given, together with the dilaton, by

$$\begin{aligned} A_1 &= \Lambda^{-1} (dx^- + c dx + (B \cdot x)^i dx_i) \\ \Phi &= \frac{3}{4} \log \Lambda , \end{aligned}$$

where

$$\Lambda = 2b + c^2 + 2V(r) + (B \cdot x)^i (B \cdot x)_i .$$

As indicated in Table 3, whenever $\theta_i \neq 0$ for all i , the two extra translation parameters must satisfy the bound $2b + c^2 > -\mu^2$, where μ is given in (2.16). We do not have a physical understanding for this configuration.

On the other hand, when $\theta_i = 0$ for all i , the bound is $2b + c^2 > 0$. In the particular case of vanishing c , the configuration looks like a delocalised D0-brane, in which x^- is playing the role of a timelike coordinate after the reduction. Thus, whenever $c \neq 0$, and following similar arguments to the ones presented in [?] when dealing with similar reductions, one could interpret the corresponding background as a bound state of type IIA waves and delocalised D0-branes. Thus, by turning on different θ parameters, we are adding fluxbranes to these bound states, and thus breaking further supersymmetry. The possible supersymmetric configurations are summarised in Table 5.

ν	Object	Subalgebra
$\frac{1}{4}$	(WA+D0)+F5	$\mathfrak{su}(2)$
$\frac{1}{8}$	(WA+D0)+F3	$\mathfrak{su}(3)$
$\frac{1}{8}$	(WA+D0)+F1	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$
$\frac{1}{16}$	(WA+D0)+F1	$\mathfrak{su}(4)$

TABLE 5. Supersymmetric configurations of bound states made of type IIA waves and delocalised D0-branes (WA+D0) and fluxbranes

3. SUPERSYMMETRIC REDUCTIONS OF THE KALUZA–KLEIN MONOPOLE

In this section we classify the supersymmetric reductions of the eleven-dimensional Kaluza–Klein monopole: a half-BPS purely gravitational M-theory background isometric to the product of the seven-dimensional Minkowski spacetime with a non-compact four-dimensional hyperkähler space: the Taub–NUT space.

The metric is given by [?, ?]

$$g = ds^2(\mathbb{E}^{1,6}) + g_{\text{TN}} , \quad (3.1)$$

where g_{TN} is the Taub–NUT metric to be described below. The isometry group is

$$G = \text{ISO}(1, 6) \times \text{U}(2) \subset \text{ISO}(1, 6) \times \text{SO}(4) \subset \text{ISO}(1, 10) , \quad (3.2)$$

with Lie algebra

$$\mathfrak{g} = (\mathbb{R}^{1,6} \rtimes \mathfrak{so}(1, 6)) \times \mathfrak{su}(2) \times \mathfrak{u}(1) . \quad (3.3)$$

Since F_4 also vanishes for this background, the Killing spinors are the parallel spinors relative to the spin connection, just as for the M-waves discussed in the previous section. Since the background is metrically a product and one factor is flat, the parallel (complexified) spinors are given by the tensor products $\eta \otimes \varepsilon$, where η is a parallel spinor of Minkowski spacetime and ε is a parallel spinor in Taub–NUT. As shown in Appendix B, ε is given by equation (B.2) where ε obeys (B.3) and hence has positive chirality. Counting dimensions we see that the Kaluza–Klein monopole is indeed a half-BPS background [?].

3.1. The Taub–NUT geometry. We shall next briefly discuss the geometry of the Taub–NUT space. (For a review see, e.g., [?].) The Taub–NUT metric is given by

$$g_{\text{TN}} = V ds^2(\mathbb{E}^3) + V^{-1}(d\chi + A)^2, \quad (3.4)$$

where the function $V : \mathbb{E}^3 \rightarrow \mathbb{R}$ and the gauge field A are related by the abelian monopole equation

$$F_A := dA = -\star dV,$$

where \star is the Hodge star in \mathbb{E}^3 . It follows from this equation that V is harmonic and that F_A obeys Maxwell's equations $d\star F_A = 0$. Although it is possible to consider multicentred solutions, we will consider for simplicity the maximally symmetric case

$$V(r) = 1 + \frac{Q}{r}, \quad \text{for some } Q > 0,$$

where r is the euclidean radius in \mathbb{E}^3 . The corresponding solution of Maxwell's equations is the Dirac monopole. In spherical polar coordinates (r, θ, φ) for \mathbb{E}^3 , where the metric is given by

$$ds^2(\mathbb{E}^3) = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

and the orientation by

$$\text{dvol}(\mathbb{E}^3) = r^2 \sin \theta dr \wedge d\theta \wedge d\varphi,$$

the gauge field can be chosen to be

$$A = -Q \cos \theta d\varphi. \quad (3.5)$$

The field-strength is proportional to the volume form on the unit sphere in \mathbb{E}^3 :

$$F_A = Q \sin \theta d\theta \wedge d\varphi,$$

whence the charge of monopole is given by

$$\frac{1}{4\pi} \int_{S^2} F_A = Q.$$

The Taub–NUT metric is therefore defined on the total space of the circle bundle over \mathbb{E}^3 (minus the origin) corresponding to a Dirac monopole of charge Q at the origin. Restricted to the unit sphere (and hence to any sphere) in \mathbb{E}^3 , this circle bundle is the Hopf fibration with total space a 3-sphere. These 3-spheres are the orbits under the isometry group $U(2) \subset SO(4)$ of the Taub–NUT metric, whence we see that it acts with cohomogeneity one. More precisely, the orbits are parametrised by $r \geq 0$. The generic orbits, which occur for $r > 0$, are 3-spheres, and there is a degenerate orbit at $r = 0$ consisting of a point: the *nut*, with due apologies to Newman, Unti and Tamburino. The foliation of the Taub–NUT space (minus the nut) by 3-spheres is analogous to the foliation of \mathbb{R}^4 (minus the origin) by 3-spheres: the difference is that whereas the spheres in \mathbb{R}^4 are round with isometry group $SO(4)$, the spheres in Taub–NUT are squashed with isometry group $U(2) \subset SO(4)$. A manifestation of this fact is that asymptotically (as $r \rightarrow \infty$),

$$g_{\text{TN}} \sim dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 + (d\chi - Q \cos \theta d\varphi)^2,$$

whence the Taub–NUT metric has a circle which remains of constant size, hence squashing the sphere, instead of growing as it would have to in order to keep the sphere round.

We can rewrite the Taub–NUT metric so that the isometries are manifest. This is made easier by identifying the orbits with the Lie group $SU(2)$ on which we have a natural action of $SU(2) \times SU(2)$ by left and right translations. The centre acts

trivially, whence we have an action of the quotient $\mathrm{SO}(4)$ which restricts to the action of the subgroup $\mathrm{U}(2)$. Let

$$\begin{aligned}\sigma_1 &= -\cos\psi d\theta + \sin\theta \sin\psi d\varphi \\ \sigma_2 &= -\sin\psi d\theta - \sin\theta \cos\psi d\varphi \\ \sigma_3 &= d\psi - \cos\theta d\varphi ,\end{aligned}\tag{3.6}$$

denote the right-invariant Maurer–Cartan forms in the Lie group $\mathrm{SU}(2)$. The range of the angular coordinates are $0 \leq \theta \leq \pi$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and $\psi \in \mathbb{R}/4\pi\mathbb{Z}$. One checks that

$$d\sigma_1 = \sigma_2 \wedge \sigma_3 , \quad d\sigma_2 = \sigma_3 \wedge \sigma_1 \quad \text{and} \quad d\sigma_3 = \sigma_1 \wedge \sigma_2 . \tag{3.7}$$

Identifying χ with $Q\psi$, we can rewrite the Taub–NUT metric (3.4) as

$$g_{\mathrm{TN}} = V dr^2 + V r^2 (\sigma_1^2 + \sigma_2^2) + V^{-1} Q^2 \sigma_3^2 . \tag{3.8}$$

Because it is written using the Maurer–Cartan forms, the invariance under $\mathrm{SU}(2)$ is manifest. There is an additional $\mathrm{U}(1)$ symmetry because of the fact that the coefficients of σ_1^2 and σ_2^2 coincide.

In the limit as $r \rightarrow 0$, the Taub–NUT metric becomes

$$g_{\mathrm{TN}} \sim \frac{Q}{r} dr^2 + Qr (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) .$$

Changing coordinates to $\varrho = 2\sqrt{Qr}$, we obtain a more familiar metric

$$g_{\mathrm{TN}} \sim d\varrho^2 + \frac{1}{4}\varrho^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) ,$$

which is the flat metric for \mathbb{R}^4 , thought of as \mathbb{C}^2 with complex coordinates

$$\begin{aligned}z_1 &= x_1 + ix_2 = \varrho \cos(\theta/2) e^{i(\psi+\varphi)/2} \\ z_2 &= x_3 + ix_4 = \varrho \sin(\theta/2) e^{i(\psi-\varphi)/2} .\end{aligned}\tag{3.9}$$

For our chosen range of coordinates $\varrho \geq 0$, $\theta \in [0, \pi]$, $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ and $\psi \in \mathbb{R}/4\pi\mathbb{Z}$, the above parametrisation covers \mathbb{R}^4 once. We conclude that the Taub–NUT metric is regular (and flat) at the nut and hence can be extended to all of \mathbb{R}^4 .

The Taub–NUT Killing vectors can be explicitly calculated as follows. From the expression for the Taub–NUT metric (3.4) it follows that ∂_χ (equivalently ∂_ψ) is a Killing vector, in fact it generates translations along the Hopf fibre. The euclidean metric in \mathbb{E}^3 and the function V are invariant under rotations in \mathbb{E}^3 , but the gauge field A is not invariant and hence they are not isometries. This can be easily fixed. These rotations do leave invariant the field-strength F_A , and hence they leave the gauge field invariant up to a compensating gauge transformation. In other words, we can modify the rotation Killing vectors in \mathbb{E}^3 by a suitable gauge transformation (i.e., a translation along the fibre) in such a way that the connection one-form $d\chi + A$ is invariant. Explicitly, let ρ be a generic rotation Killing vector in \mathbb{E}^3 . The fact that F_A is invariant means that

$$\mathcal{L}_\rho dA = d\mathcal{L}_\rho A = 0 \implies \mathcal{L}_\rho A = d\Lambda_\rho ,$$

for some function Λ_ρ . This means that A is invariant up to a gauge transformation, whence $\xi := \rho - \Lambda_\rho \partial_\chi$ leaves invariant $d\chi + A$ and hence the Taub–NUT metric. Notice that Λ_ρ is only defined up to a constant. We choose this constant in order that the Killing vectors obey the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ algebra. In terms of the orbit

coordinates (θ, φ, ψ) , the Killing vectors are given by

$$\begin{aligned}\xi_1 &= -\sin \varphi \partial_\theta - \cot \theta \cos \varphi \partial_\varphi - \frac{\cos \varphi}{\sin \theta} \partial_\psi \\ \xi_2 &= -\cos \varphi \partial_\theta + \cot \theta \sin \varphi \partial_\varphi + \frac{\sin \varphi}{\sin \theta} \partial_\psi \\ \xi_3 &= \partial_\varphi \\ \xi_4 &= \partial_\psi .\end{aligned}\tag{3.10}$$

One can also check directly that the first three such vectors leave the σ_i invariant, whereas the fourth rotates σ_1 and σ_2 , which provides an alternative proof that they leave the metric invariant. In fact, the first three are the left-invariant vector fields on the Lie group $SU(2)$ and, as can be easily checked, satisfy $[\xi_i, \xi_j] = \varepsilon_{ijk} \xi_k$ for $i, j, k = 1, 2, 3$. The remaining vector field ξ_4 is right-invariant and commutes with the other three. In other words, these Killing vectors define a realisation of $\mathfrak{su}(2) \times \mathfrak{u}(1)$, with $\xi_{1,2,3}$ spanning $\mathfrak{su}(2)$ and ξ_4 spanning $\mathfrak{u}(1)$.

3.2. Freely-acting spacelike isometries. Our main physical motivation is to study IIA configurations involving D6-branes and flux- and nullbranes, but to obtain the D6-brane in type IIA we need to reduce the Kaluza–Klein monopole along the Hopf fibre of the Taub–NUT space. This is generated by the vector field ξ_4 above which vanishes at the nut. In fact, upon reduction, the IIA solution has a naked singularity at the nut, which is where the D6-branes lie. Since we are interested in generalising this reduction in order to incorporate fluxbranes, we will allow for isometries which are null when $r = 0$, but spacelike everywhere else:

$$\|\xi\|^2 \geq 0 \quad \text{and} \quad \|\xi\|^2 > 0 \quad \text{for } r > 0.$$

This is analogous to allowing isometries of brane backgrounds which are spacelike everywhere but at the brane horizon, as we did in [?]. We beg the reader's indulgence in allowing us the slight abuse of notation in referring to these Killing vectors as *spacelike* within the confines of this section.

It follows from the structure (3.3) of the Lie algebra of isometries of the Kaluza–Klein monopole, that the most general infinitesimal isometry can be written as

$$\xi = \tau + \lambda + \rho_{\text{TN}} ,$$

where $\tau \in \mathbb{R}^{1,6}$, $\lambda \in \mathfrak{so}(1,6)$ and $\rho_{\text{TN}} \in \mathfrak{su}(2) \times \mathfrak{u}(1)$. Notice that ρ_{TN} is orthogonal to $\tau + \lambda$ and its norm is positive-definite except at $r = 0$, where it vanishes. We can use the freedom to conjugate by G in order to bring these Killing vectors to a normal form. We will treat both factors separately.

First let us consider the Taub–NUT factor. Conjugating by $SU(2)$ allows us to bring ρ_{TN} to the form

$$\rho_{\text{TN}} = a \partial_\psi + b \partial_\varphi ,$$

for some constants a, b . The norm of this vector field is positive away from the nut provided that $a \pm b \neq 0$. Indeed,

$$\|\rho_{\text{TN}}\|^2 = b^2 V(r) \sin^2 \theta + V(r)^{-1} Q^2 (a - b \cos \theta)^2 ,$$

which is clearly positive for $r > 0$ unless $a = \pm b$.

The analysis of the Minkowski factor is similar to the ones given in [?] for the M2-brane and M5-brane backgrounds. Conjugating by an isometry, we may bring the Lorentz component to one of the following normal forms:

1. $\lambda = \theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56}$;
2. $\lambda = \beta B_{01} + \theta_2 R_{34} + \theta_3 R_{56}$, $\beta \neq 0$; or
3. $\lambda = N_{+2} + \theta_2 R_{34} + \theta_3 R_{56}$,

where B_{0i} and N_{+i} stand, respectively, for an infinitesimal boost and null rotation in the i th direction. Conjugating by the translation subgroup $\mathbb{R}^{1,6}$, we can bring the translation to a normal form depending on the form of λ . In case (1), with none of the θ_i vanishing, the translation can be made proportional to ∂_0 ; but then the norm of $\tau + \lambda$ would be negative at some points, unless $\tau = 0$. If (at least) one of the θ_i were to vanish, say $\theta_1 = 0$, then τ can be taken to be any translation in the (01) plane. Its norm cannot be negative, otherwise there would be points where ξ would have negative norm: this means that τ can be either spacelike or null. In case (2), we can take τ proportional to ∂_2 ; but the boost would make ξ have negative norm at points. This case is therefore discarded. Finally, in case (3), if none of the θ_i vanish, τ can be taken to be proportional to ∂_- , but then ξ would have negative norm at some points unless $\tau = 0$. On the other hand, if one of the θ_i vanish, say $\theta_2 = 0$, then we can take τ proportional to ∂_3 .

In contrast to the M-wave and the backgrounds discussed in [?], in the Kaluza–Klein monopole there are spacelike Killing vectors which generate circle actions instead of \mathbb{R} -actions. Such Killing vectors have no proper translations and hence the analysis of whether they give rise to smooth reductions is more delicate than in previous backgrounds. To see what can go wrong, simply notice that the integral curves of the vector field $a\partial_\psi + b\partial_\varphi$ away from the nut in Taub–NUT lie generically in a torus. If these curves fail to be periodic (that is, if the ratio a/b is not rational), the corresponding reduction would not even be Hausdorff. As we will see, this will manifest itself in reductions which have no continuous moduli.

Therefore we will distinguish between two types of spacelike Killing vectors $\xi = \tau + \lambda + \rho_{\text{TN}}$, according to whether τ vanishes or not. From the observations made above, those with nonzero τ fall into three cases:

- (A) $\xi = \partial_+ + \theta_2 R_{34} + \theta_3 R_{56} + a\partial_\psi + b\partial_\varphi$, where $a \neq \pm b$. Such vector fields comprise a four-dimensional stratum of the moduli space of smooth reductions.
- (B) $\xi = \partial_1 + \theta_2 R_{34} + \theta_3 R_{56} + a\partial_\psi + b\partial_\varphi$. The corresponding stratum is four-dimensional.
- (C) $\xi = \partial_3 + N_{+2} + \theta_3 R_{56} + a\partial_\psi + b\partial_\varphi$. This stratum is three-dimensional.

Notice that we have already used the freedom to rescale the Killing vector. In all of these cases, the action of ξ integrates to a free action of \mathbb{R} . We will see below that supersymmetry will select a codimension-one locus.

Similarly, there are two cases with $\tau = 0$:

- (a) $\xi = \theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + a\partial_\psi + b\partial_\varphi$; and
- (b) $\xi = N_{+2} + \theta_2 R_{34} + \theta_3 R_{56} + a\partial_\psi + b\partial_\varphi$,

and these require further attention. We can easily discard case (b) by an argument similar to that employed for the reduction of the M2 brane which we labelled (C) in [?, Section 3.1.1]. Although the action is free, there are two types of orbits: at those points where the Killing vector corresponding to the null rotation vanishes, the orbits are circles, whereas at other points they are real lines. This means that what we have is an action of \mathbb{R} with nontrivial stabilisers at those points where the orbits are closed. As a result the quotient is not smooth.

The vector field ξ in (a) is a vector field on $\mathbb{R}^6 \times S^3 \subset \mathbb{R}^{10}$. It is convenient to identify \mathbb{R}^{10} with \mathbb{C}^5 and introduce complex coordinates $(w_1, w_2, w_3, z_1, z_2)$ where z_1 and z_2 are given by equation (3.9). The integrated action of ξ on \mathbb{C}^5 is given by

$$(w_1, w_2, w_3, z_1, z_2) \mapsto \left(e^{i\theta_1 t} w_1, e^{i\theta_2 t} w_2, e^{i\theta_3 t} w_3, e^{i(a+b)t/2} z_1, e^{i(a-b)t/2} z_2 \right).$$

Generically this defines a curve in the 5-tori defined by fixing the values of $|w_i|$ and $|z_i|$. Away from the nut we must have that at least one of z_i is nonzero, whereas the w_i are allowed to vanish. Unless the integral curves are periodic, the quotient of such a torus will not be Hausdorff: this is essentially the same situation as the

more familiar irrational flows on the 2-torus. To avoid this situation, there must be some $T > 0$ for which the phase factors

$$e^{i\theta_i T}, \quad e^{i(a+b)T/2} \quad \text{and} \quad e^{i(a-b)T/2}$$

must all be equal to 1; equivalently,

$$\theta_i T = 2\pi p_i, \quad aT = 2\pi(n+m) \quad \text{and} \quad bT = 2\pi(n-m),$$

for some integers p_i, m, n . Taking T to be the period (the smallest positive number with this property), we must have that $\gcd(p_1, p_2, p_3, n+m, n-m) = 1$. In any case, the ratio of any two of θ_i, a, b is rational whenever it is defined.

By considering the points where $w_i = 0$, we notice that in order to have a free action, a and b cannot both vanish. Therefore we must distinguish between two cases: $a \neq 0$ and $a = 0$ (whence $b \neq 0$). In the first case, $n \neq -m$, so we can rewrite the Killing vector as

$$\xi = \frac{a}{n+m} (p_1 R_{12} + p_2 R_{34} + p_3 R_{56} + (n+m)\partial_\psi + (n-m)\partial_\varphi).$$

Using the freedom to rescale ξ , we arrive at

$$\xi = p_1 R_{12} + p_2 R_{34} + p_3 R_{56} + (n+m)\partial_\psi + (n-m)\partial_\varphi,$$

where $\gcd(p_1, p_2, p_3, n+m, n-m) = 1$. Its integral curves are given by

$$(e^{ip_1 t} w_1, e^{ip_2 t} w_2, e^{ip_3 t} w_3, e^{int} z_1, e^{imt} z_2),$$

which have period 2π . If $a = 0$, so that $m = -n$, we can rewrite the vector field as

$$\xi = \frac{b}{2n} (p_1 R_{12} + p_2 R_{34} + p_3 R_{56} + 2n\partial_\varphi).$$

We can again rescale to arrive at

$$\xi = p_1 R_{12} + p_2 R_{34} + p_3 R_{56} + 2n\partial_\varphi,$$

where $\gcd(p_1, p_2, p_3, 2n) = 1$, which integrates to

$$(e^{ip_1 t} w_1, e^{ip_2 t} w_2, e^{ip_3 t} w_3, e^{int} z_1, e^{-int} z_2),$$

which again has period 2π . Notice that both cases reduce to studying the orbits

$$(e^{ip_1 t} w_1, e^{ip_2 t} w_2, e^{ip_3 t} w_3, e^{int} z_1, e^{imt} z_2),$$

where $\gcd(p_1, p_2, p_3, n-m, n+m) = 1$, where n and m need not be different, but cannot both be zero.

To obtain a smooth quotient the stabilisers of all points (away from the nut) must be trivial. This means that when z_1 and z_2 are not both zero, the only solution to

$$(w_1, w_2, w_3, z_1, z_2) = (e^{ip_1 t} w_1, e^{ip_2 t} w_2, e^{ip_3 t} w_3, e^{int} z_1, e^{imt} z_2),$$

must be $t \in 2\pi\mathbb{Z}$. Considering the points $(0, 0, 0, 0, 1)$ and $(0, 0, 0, 1, 0)$ we see that this is the case if and only if $n = \pm 1$ and $m = \pm 1$, giving four cases in total: the two cases where $n = m$ correspond to $b = 0$ and the other two cases correspond to $a = 0$. Changing the sign of ξ , if necessary, which is the only rescaling freedom left, we can choose $n = 1$. This gives two cases:

- (i) $\xi = 2\partial_\psi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$, and
- (ii) $\xi = 2\partial_\varphi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$,

where $p_i \in \mathbb{Z}$ and $\gcd(2, p_1, p_2, p_3) = 1$. It is easy to see that there are no further conditions on the p_i .

In summary, there are two possible cases of freely-acting spacelike Killing vectors without translations and hence with integer moduli p_i :

- (D) $\xi = 2\partial_\psi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$, with $\gcd(2, p_1, p_2, p_3) = 1$;
- (E) $\xi = 2\partial_\varphi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$, with $\gcd(2, p_1, p_2, p_3) = 1$;

Notice that these cases define genuine circle actions, without the need to further identify points in spacetime.

3.3. Supersymmetry. The Killing spinors in a purely gravitational background are precisely the parallel spinors with respect to the spin connection. For the Kaluza–Klein monopole these are tensor products of parallel spinors in Minkowski spacetime with parallel spinors in the Taub–NUT space. In the flat coordinates for Minkowski spacetime, parallel spinors are simply constant spinors in the half-spin representation of $\text{Spin}(1, 6)$. The parallel spinors in the Taub–NUT space are computed explicitly in Appendix B. The result is that parallel spinors are in one-to-one correspondence with positive-chirality spinors for $\text{Spin}(4)$. Moreover this correspondence is equivariant with respect to the action of (the spin cover of) the isometry group: $\text{SU}(2) \times \text{U}(1)$. This allows us to easily determine the constraints imposed by supersymmetry on the reductions classified in the previous section. As usual, translations act trivially on spinors and a null rotation simply halves the number of invariant spinors, so it is only the rotational component which is constrained.

As explained in Appendix A, the Killing spinors of the Kaluza–Klein monopole are in one-to-one correspondence with the subspace of the eleven-dimensional spinor representation S_{11} which, under $\text{Spin}(1, 6) \times \text{SU}(2) \times \text{U}(1)$, transforms as $[S_7 \otimes S_3]$, where S_7 is the unique irreducible spinor representation of $\text{Spin}(1, 6)$, S_3 is the fundamental of $\text{SU}(2)$ and $\text{U}(1)$ acts trivially. This $\text{U}(1)$ is generated by the Hopf “translation” ∂_ψ , although one should keep in mind that from the flat space point of view, this is a self-dual rotation and certainly acts on spinors in that way. It is the fact that the spinors have positive chirality which makes ∂_ψ act trivially on them.

Let us first classify the supersymmetric reductions with translations, and hence with continuous moduli. Let ξ be a freely-acting spacelike Killing vector with rotational component

$$\rho = \theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + b \partial_\varphi ,$$

where θ_i and b are real numbers. We have not included the rotation $a \partial_\psi$ in ρ since this acts trivially on spinors. The action of ρ on $[S_7 \otimes S_3]$ can be read off from the weight decomposition (A.2).¹ Supersymmetry will be preserved if and only if

$$\mu_1 \theta_1 + \mu_2 \theta_2 + \mu_3 \theta_3 + \mu_4 b = 0 ,$$

where $\mu_i^2 = 1$. These equations define a collection of four hyperplanes in the four-dimensional space parametrised by (θ_i, b) . If ρ belongs to precisely one of these hyperplanes, it is annihilated by precisely two weights in $[S_7 \otimes S_3]$. If ρ belongs to the intersection of precisely two hyperplanes, then it is annihilated by four weights. If in the intersection of precisely three hyperplanes, then it is annihilated by eight weights. Finally the only point in the intersection of more than three such hyperplanes is the origin which is annihilated by all sixteen weights.

After these preliminary observations it is easy to read off the different strata of the “continuous” moduli space of supersymmetric reductions of the Kaluza–Klein monopole. Using the same nomenclature for the possible Killing vectors, we find the following cases, which are summarised in Table 6.

- (A) In this case the rotation ρ has $\theta_1 = 0$ and hence must belong to the intersection of an $\mathfrak{su}(3)$ and an $\mathfrak{so}(4) \times \mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(8)$. The resulting three-dimensional stratum has $\nu = \frac{1}{8}$. There is supersymmetry enhancement to $\nu = \frac{1}{4}$ in the two-dimensional locus corresponding to rotations ρ which belong to the intersection of an $\mathfrak{su}(2)$ and an $\mathfrak{so}(4) \times \mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(8)$. In

¹In the notation of Appendix A, ∂_φ can be thought of as R_{78} .

addition there is a one-dimensional locus, corresponding to vanishing ρ , where the supersymmetry is enhanced to $\nu = \frac{1}{2}$.

- (B) This is the same as (A).
- (C) In this case we have $\theta_1 = \theta_2 = 0$. We therefore have a two-dimensional stratum of supersymmetric reductions with $\nu = \frac{1}{8}$ and consisting of rotations ρ in the intersection of an $\mathfrak{su}(2)$ and an $\mathfrak{so}(4) \times \mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(6)$. There is a one-dimensional sublocus consisting of vanishing ρ , where supersymmetry is enhanced to $\nu = \frac{1}{4}$.

Translation	Subalgebra	ν	dim
∂_+ or ∂_y	$\cap \mathfrak{su}(3)$	$\frac{1}{8}$	3
	$\cap \mathfrak{su}(2)$	$\frac{1}{4} \quad (\frac{1}{8})$	2 (2)
	$\{0\}$	$\frac{1}{2} \quad (\frac{1}{4})$	1 (1)

TABLE 6. Supersymmetric reductions of the Kaluza–Klein monopole with continuous moduli. The notation $\cap \mathfrak{h}$ in the “Subalgebra” column indicates that the subalgebra is the intersection of \mathfrak{h} with the rotational subalgebra of the isometry algebra. A ∂_y means a spacelike translation along a Minkowski direction. The numbers in parentheses indicate the values in the presence of a null rotation, which can only happen when the translation is spacelike.

We now move on to discuss the supersymmetric reductions without translations, and hence with only discrete moduli. In this case, as the action is that of a circle, which is not simply-connected, there exists the question of whether the quotient admits a spin structure. As shown in Appendix C for the reductions labelled (D) and (E), the quotient admits a spin structure if and only if the sum of the p_i is even. We will therefore assume from now on that this is the case. Notice that since the p_i cannot all be even, precisely two of them must be odd.

Instead of hyperplanes in the continuous moduli space, supersymmetry now imposes linear diophantine equations on the integer moduli of the reductions. A similar analysis to the one before, but paying attention to the fact that the p_i are integers which cannot all be even, yields the following results, which are summarised in Table 7.

- (D) In this case the rotation is $\rho = p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$. Supersymmetry imposes the linear diophantine equation

$$p_1 \mu_1 + p_2 \mu_2 + p_3 \mu_3 = 0 .$$

There are clearly an infinite number of solutions to this equation for which precisely two of the p_i are odd. The rotation is contained in an $\mathfrak{su}(3)$ subalgebra of $\mathfrak{so}(6)$ and the reduction preserves a fraction $\nu = \frac{1}{8}$ of the supersymmetry. There is supersymmetry enhancement to $\nu = \frac{1}{4}$ whenever precisely one of the p_i vanishes. The nonzero two p_i must then be odd integers. Clearly there are again an infinite number of such solutions. In addition there is a unique reduction with $\nu = \frac{1}{2}$ corresponding to (a stack of) D6-branes.

- (E) Here the rotation ρ takes the form $\rho = p_1 R_{12} + p_2 R_{34} + p_3 R_{56} + 2\partial_\varphi$. Supersymmetry imposes the equation (after some relabelling)

$$p_1 \mu_1 + p_2 \mu_2 + p_3 \mu_3 = 2 .$$

There are clearly an infinite number of solutions with precisely two of the p_i odd. The generic reduction preserves a fraction $\nu = \frac{1}{16}$ of the supersymmetry. The rotation belongs to the intersection of an $\mathfrak{su}(4)$ and an $\mathfrak{so}(6) \times \mathfrak{su}(2)$ subalgebras of $\mathfrak{so}(10)$. There is supersymmetry enhancement to $\nu = \frac{1}{8}$ in either of two situations: when precisely one of the p_i vanishes, which corresponds to the intersection of an $\mathfrak{su}(3)$ subalgebra with an $\mathfrak{so}(6) \times \mathfrak{su}(2)$ subalgebra of $\mathfrak{so}(10)$; and when the equation decouples into two equations: $p_1 = \pm p_2$ and $p_3 = \pm 2$, say, with p_1 and p_2 odd. This corresponds to intersecting an $\mathfrak{sp}(1) \times \mathfrak{sp}(1)$ subalgebra. One might expect further supersymmetry enhancement by intersecting an $\mathfrak{su}(2)$ subalgebra, but this would require two of the p_i to vanish and then the remaining nonzero $p = \pm 2$ would not be odd. Therefore no further enhancement takes place.

Although it appears from the above discussion that the D6-brane is isolated, it actually lives in the same moduli space as case (A) or (B) with vanishing ρ . In fact, those reductions are all special cases of reductions by a linear system $a\partial_\psi + b\partial_1 + c\partial_+$ of Killing vectors. The D6-brane corresponds to $b = c = 0$, but one can clearly deform it without sacrificing supersymmetry by turning on a minkowskian translation which can be either spacelike or null.

ρ_{TN}	Subalgebra	ν
$2\partial_\psi$	$\mathfrak{su}(3)$	$\frac{1}{8}$
	$\mathfrak{su}(2)$	$\frac{1}{4}$
	$\{0\}$	$\frac{1}{2}$
$2\partial_\varphi$	$\cap \mathfrak{su}(4)$	$\frac{1}{16}$
	$\cap \mathfrak{sp}(1) \times \mathfrak{sp}(1)$	$\frac{1}{8}$
	$\cap \mathfrak{su}(3)$	$\frac{1}{8}$

TABLE 7. Supersymmetric reductions of the Kaluza–Klein monopole without continuous moduli. The notation $\cap \mathfrak{h}$ in the “Subalgebra” column indicates that the subalgebra is the intersection of \mathfrak{h} with the rotational subalgebra of the isometry algebra.

3.4. Explicit reductions. We shall start by studying the reductions denoted (B) and (C) in Section 3.3. We should mention that some of the configurations described in this section have some overlap with the content of the paper [?]. The Killing vector can be written as $\xi = \partial_z + \lambda$, where z stands for a longitudinal direction, e.g. y^1 , and λ stands for the infinitesimal transformation

$$\begin{aligned} \lambda = & \beta(y^0\partial_3 + y^3\partial_0) + \theta_2(y^3\partial_4 - y^4\partial_3) + \theta_3(y^5\partial_6 - y^6\partial_5) \\ & + a\partial_\psi + b\partial_\varphi . \end{aligned}$$

The constant matrix B defined in (1.2) is a 7×7 matrix which can be written as

$$B = \begin{pmatrix} 0 & \beta & 0 & 0 & 0 & 0 & 0 \\ \beta & 0 & -\theta_2 & 0 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & 0 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (3.11)$$

in the basis of adapted coordinates (1.1) $\{x^0, x^3, x^4, x^5, x^6, \psi, \varphi\}$. Notice that besides z , it does not act on $\{x^2, r, \theta\}$. Since λ involves $a\partial_\psi + b\partial_\varphi$, there is a nontrivial \mathbf{C} vector, which in the same basis used for (3.11) is written as

$$(\mathbf{C})^t = (\vec{0}, a, b).$$

Since the starting configuration is a purely gravitational background, the type IIA configuration involves a ten dimensional metric, dilaton and a RR 1-form potential, all other fields vanishing. We shall introduce a new notation for the RR 1-form potential C_1 , to avoid any confusion with the 1-form potential describing the Dirac monopole in (3.5). The full configuration looks like

$$\begin{aligned} g &= \Lambda^{1/2} \{ds^2(\mathbb{E}^{1,5}) + g_{\text{TN}}\} - \Lambda^{3/2} (C_1)^2 \\ C_1 &= \Lambda^{-1} \{(B\mathbf{x})^i (d\mathbf{x})_i + bV(r)r^2 \sin^2 \theta d\varphi \\ &\quad + Q^2 V^{-1}(r)(a - b \cos \theta)(d\psi - \cos \theta d\varphi)\} \\ \Phi &= \frac{3}{4} \log \Lambda, \end{aligned} \quad (3.12)$$

where $(d\mathbf{x})_i = \delta_{ij} d\mathbf{x}^j$. It depends on an scalar function Λ which is defined as

$$\Lambda = 1 + (B\mathbf{x})^i (B\mathbf{x})_i + b^2 V(r)r^2 \sin^2 \theta + Q^2 V^{-1}(r)(a - b \cos \theta)^2.$$

Notice that whenever there are fluxbranes being described in ten dimensions, $\theta_i \neq 0$, $b \neq 0$, there are regions in spacetime where the string coupling constant becomes strong, and so the type IIA supergravity description is no longer reliable.

As discussed in Appendix B.2, the Killing vector ∂_φ acts as a rotation inside $\mathfrak{su}(2)$, and as such it should be treated at the same level as the other generators of rotations in the flat directions along the monopole. Having remarked this point, the interpretation of the different points in the moduli space can be given as follows. If all parameters are set to zero, (3.12) describes a ten dimensional Kaluza–Klein monopole. Whenever $a \neq 0$, and following the same arguments used in [?], the interpretation is that of a bound state of a Kaluza-Klein monopole and D6-branes [?]. Before discussing the other possibilities, it is interesting to analyse this bound state closer. From the parametrisation (3.9), it is evident that locally $\partial_\psi \propto R_{78} + R_{9\bar{9}}$. Thus, the reduction looks very similar to the one defining a flux 5-brane [?]. Actually both preserve the same supersymmetries, and by considering the limit $r \rightarrow 0$ in (3.12), one is left with the metric and RR 1-form

$$\begin{aligned} g &= \Lambda^{1/2} \left\{ ds^2(\mathbb{E}^{1,5}) + \frac{Q}{r} ((dr)^2 + r^2((d\theta)^2 + \sin^2 \theta (d\varphi)^2)) \right\} \\ &\quad + \Lambda^{-1/2} r \cdot Q (d\psi - \cos \theta d\varphi)^2 \\ C_1 &= a r \cdot Q \Lambda^{-1} (d\psi - \cos \theta d\varphi), \end{aligned}$$

depending on the scalar function $\Lambda = 1 + a^2 r \cdot Q$. By changing the radial coordinate, $\hat{r} = 2\sqrt{Q \cdot r}$, one recovers the flux 5-brane first introduced in [?] and

identifies the arbitrary parameter a in the above construction, with the parameter β characterising the fluxbrane “charge” through the relation

$$\beta = \pm \frac{a}{2}.$$

One can thus conclude that the flux 5-brane describes the local region (close to the branes) of a bound state of Kaluza-Klein monopoles and D6-branes.

By switching on the remaining parameters, we are adding the corresponding fluxbranes and nullbranes to the configuration. Thus whenever $|\beta| = |\theta_2|$, there will be a nullbrane. In that case, we still have the possibility to add a flux 5-brane (F5-brane), whenever $\theta_3 \pm b = 0$. Setting $\beta = 0$, there are three different fluxbranes that one can construct:

1. A flux 3-brane (F3-brane) when $\theta_1 \pm \theta_2 \pm b = 0$.
2. A F5-brane when $\theta_2 \pm \theta_3 = 0$.
3. Due to the isometries of the background there is a further F5-brane when $\theta_1 \pm b = 0$.

All other possibilities would break supersymmetry, and as such, they can be interpreted in terms of F7-branes (or intersections thereof with no supersymmetry enhancement) or as quotients by the orbits of boosts [?, ?, ?, ?]. In the latter case, there would be regions of spacetime with closed timelike curves. The supersymmetric configurations are summarised in Tables 8 and 9. We use the notation Fp' to denote those fluxbranes which involve the rotation in the 3-sphere foliating the original Taub–NUT space.

ν	Object	Subalgebra
$\frac{1}{4}$	KK+N	\mathbb{R}
$\frac{1}{4}$	KK \perp F5	$\cap \mathfrak{su}(2)$
$\frac{1}{4}$	KK \perp F5'	$\cap \mathfrak{su}(2)$
$\frac{1}{8}$	KK+N+F5'	$\mathbb{R} \times \cap \mathfrak{su}(2)$
$\frac{1}{8}$	KK+F3'	$\cap \mathfrak{su}(3)$

TABLE 8. Supersymmetric configurations of Kaluza–Klein monopoles (KK) and fluxbranes and nullbranes

ν	Object	Subalgebra
$\frac{1}{4}$	(KK-D6)+N	\mathbb{R}
$\frac{1}{4}$	(KK-D6) \perp F5	$\cap \mathfrak{su}(2)$
$\frac{1}{4}$	(KK-D6) \perp F5'	$\cap \mathfrak{su}(2)$
$\frac{1}{8}$	(KK-D6)+N+F5'	$\mathbb{R} \times \cap \mathfrak{su}(2)$
$\frac{1}{8}$	(KK-D6)+F3'	$\cap \mathfrak{su}(3)$

TABLE 9. Supersymmetric configurations of bound states of a Kaluza–Klein monopole and D6-branes (KK-D6) and fluxbranes and nullbranes

Let us now move to study the reductions referred to as (A) in Section 3.3. The Killing vector can be written as $\xi = \partial_z + \lambda$, where z stands for a lightlike direction,

i.e. y^+ , and λ stands for the infinitesimal transformation

$$\lambda = \theta_2(y^3\partial_4 - y^4\partial_3) + \theta_3(y^5\partial_6 - y^6\partial_5) + a\partial_\psi + b\partial_\varphi .$$

The constant matrix B in (1.2) is a 6×6 matrix which can be written as

$$B = \begin{pmatrix} 0 & -\theta_2 & 0 & 0 & 0 & 0 \\ \theta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta_3 & 0 & 0 \\ 0 & 0 & \theta_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \quad (3.13)$$

in the basis $\{x^3, x^4, x^5, x^6, \psi, \varphi\}$. Notice that besides z , it does not act on $\{x^-, x^2, r, \theta\}$. There is again a nontrivial \mathbf{C} vector, which in the same basis used for (3.13) is written as

$$(\mathbf{C})^t = (\vec{0}, a, b) .$$

Adopting the same notation as before, the full type IIA configuration is written as follows

$$\begin{aligned} g &= \Lambda^{1/2} \{ds^2(\mathbb{E}^5) + g_{\text{TN}}\} - \Lambda^{3/2} (C_1)^2 \\ C_1 &= \Lambda^{-1} \{dx^- + (B\mathbf{x})^i (d\mathbf{x})_i + bV(r)r^2 \sin^2 \theta d\varphi \\ &\quad + Q^2 V^{-1}(r)(a - b \cos \theta)(d\psi - \cos \theta d\varphi)\} \\ \Phi &= \frac{3}{4} \log \Lambda . \end{aligned}$$

The configuration depends on a scalar function Λ which is defined as

$$\Lambda = (B\mathbf{x})^i (B\mathbf{x})_i + b^2 V(r)r^2 \sin^2 \theta + Q^2 V^{-1}(r)(a - b \cos \theta)^2 .$$

The physical interpretation of these configurations is not clear to us. It is straightforward to derive the set of supersymmetric configurations. Either with ($a \neq 0$) or without D6-branes ($a=0$), we can add a F3-brane' when $\theta_2 \pm \theta_3 \pm b =$ and a F5-brane' when $\theta_2 \pm b = 0$. Since $a \neq \pm b$, when $a \neq 0$, we can also add a standard F5-brane for $\theta_2 \pm \theta_3 = 0$.

Finally, we shall analyse the reductions involving discrete moduli. To begin with, we shall discuss the case referred to as (D) in Section 3.3. The Killing vector can be written as $\xi = \partial_z + \lambda$, where z stands for the compact coordinate along the Hopf fibre, i.e. ψ , and λ stands for the infinitesimal transformation

$$\lambda = p_1(y^1\partial_2 - y^2\partial_1) + p_2(y^3\partial_4 - y^4\partial_3) + p_3(y^5\partial_6 - y^6\partial_5) .$$

The constant matrix B in (1.2) is a 6×6 matrix

$$B = \begin{pmatrix} 0 & -p_1 & 0 & 0 & 0 & 0 \\ p_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -p_2 & 0 & 0 \\ 0 & 0 & p_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -p_3 \\ 0 & 0 & 0 & 0 & p_3 & 0 \end{pmatrix} , \quad (3.14)$$

in the basis $\{x^1, \dots, x^6\}$. Notice that it does not act on the Taub–NUT space.

After the Kaluza–Klein reduction, the full type IIA configuration is written as follows

$$\begin{aligned} g &= \Lambda^{1/2} \{ ds^2(\mathbb{E}^{1,6}) + V(r) ds^2(\mathbb{E}^3) + Q^2 V^{-1}(r) \cos^2 \theta (d\varphi)^2 \} \\ &\quad - \Lambda^{3/2} (C_1)^2 \\ C_1 &= \Lambda^{-1} \{ (B\mathbf{x})^i (d\mathbf{x})_i - 2Q^2 V^{-1}(r) \cos \theta d\varphi \} \\ \Phi &= \frac{3}{4} \log \Lambda . \end{aligned}$$

The configuration depends on an scalar function Λ which is defined as

$$\Lambda = (B\mathbf{x})^i (B\mathbf{x})_i + 4Q^2 V^{-1}(r) .$$

It should be clear that when we set all parameters to zero, this gives raise to the standard half-BPS D6-branes. By switching on different moduli, one is thus adding fluxbranes. The new feature, as discussed before, is that due to the fact that ∂_ψ gives raise to a circle action, the values of these parameters are no longer continuous. Table 10 summarises the set of supersymmetric composite configurations of D6-branes and fluxbranes.

ν	Object	Subalgebra
$\frac{1}{4}$	D6 \perp F5(2)	$\mathfrak{su}(2)$
$\frac{1}{8}$	D6 \perp F3(0)	$\mathfrak{su}(3)$

TABLE 10. Supersymmetric configurations of D6-branes and fluxbranes

There is one other case referred to as (E) in Section 3.3 involving discrete moduli. The Killing vector can be written as $\xi = \partial_z + \lambda$, where z stands for the angular coordinate φ , and λ is the same as in the previous reduction. The constant matrix B equals (3.14) and the full type IIA configuration is written as follows

$$\begin{aligned} g &= \Lambda^{1/2} \{ ds^2(\mathbb{E}^{1,6}) + V(r) ((dr)^2 + r^2 (d\theta)^2) + Q^2 V^{-1}(r) (d\psi)^2 \} - \Lambda^{3/2} (C_1)^2 \\ C_1 &= \Lambda^{-1} \{ (B\mathbf{x})^i (d\mathbf{x})_i - 2Q^2 V^{-1}(r) \cos \theta d\psi \} \\ \Phi &= \frac{3}{4} \log \Lambda . \end{aligned}$$

The configuration depends on an scalar function Λ which is defined as

$$\Lambda = (B\mathbf{x})^i (B\mathbf{x})_i + 4V(r)r^2 \sin^2 \theta + 4Q^2 V^{-1}(r) \cos^2 \theta .$$

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APPENDIX A. GROUP THEORY AND SPINORS

In this appendix we collect some facts about how the spinor representation of $\text{Spin}(1, 10)$ decomposes under certain subgroups. These results are useful in determining the supersymmetric Kaluza–Klein reductions of the M-wave and the Kaluza–Klein monopole.

Let us start by recalling a few facts about the irreducible representations of $\text{Spin}(1, 10)$ and of the Clifford algebra $\text{Cl}(1, 10)$. The Clifford algebra $\text{Cl}(1, 10)$ is isomorphic (as a real associative algebra) to $\text{Mat}(32, \mathbb{R}) \oplus \text{Mat}(32, \mathbb{R})$, where $\text{Mat}(n, \mathbb{R})$ is the algebra of $n \times n$ real matrices. This means that there are two inequivalent irreducible representations: real and of dimension 32. They are distinguished by the action of the centre which is generated by the volume form

$$\text{dvol}(\mathbb{E}^{1,10}) := \Gamma_{01\dots 11} ,$$

which squares to the identity. Let us assume that a choice has been made once and for all and let S_{11} denote the corresponding irreducible representation. This is an irreducible representation of $\text{Spin}(1, 10)$.

In order to determine which Kaluza–Klein reductions of the M-wave preserve some supersymmetry, we must decompose the subspace $\ker \Gamma_+ \subset S_{11}$ into irreducible representations of $\text{Spin}(9)$ and then determine the weight decomposition in terms of a Cartan subalgebra of $\mathfrak{so}(9)$. Let us decompose S_{11} as $S_{11} = S_{11}^+ \oplus S_{11}^-$, where $S_{11}^\pm = \ker \Gamma_\pm$. The transverse spin group $\text{Spin}(9)$ acts on S_{11} preserving the subspaces S_{11}^\pm , which are isomorphic under $\text{Spin}(9)$ to the unique irreducible spinor representation S_9 . Cartan subalgebras of $\mathfrak{so}(9)$ are actually contained in an $\mathfrak{so}(8)$ subalgebra, under which S_9 breaks up as $S_9 = S_8^+ \oplus S_8^-$, where now the label \pm refers to the eight-dimensional chirality. The weight decomposition of S_9 relative to a basis dual to $\{R_{2i-1, 2i}\}$ is given by:

$$\text{weights}(S_9) = \left\{ (\mu_1, \mu_2, \mu_3, \mu_4) \left| \mu_i^2 = 1 \right. \right\} . \quad (\text{A.1})$$

Finally, we discuss the case of the Kaluza–Klein monopole. The relevant subgroup of $\text{Spin}(1, 10)$ is now $\text{Spin}(1, 6) \times \text{SU}(2) \times \text{U}(1)$, where $\text{SU}(2) \times \text{U}(1) \subset \text{Spin}(4)$ is the “spin cover” of $\text{U}(2) \subset \text{SO}(4)$. There is a unique half-spin representation S_7 of $\text{Spin}(1, 6)$: it is quaternionic, of complex dimension 8. There are two possible actions of the Clifford algebra $\text{Cl}(1, 6)$ on S_7 distinguished by whether $\text{dvol}(\mathbb{E}^{1,6})$, which squares to $+\mathbb{1}$, acts as $\pm\mathbb{1}$. The decomposition

$$\text{dvol}(\mathbb{E}^{1,10}) = \text{dvol}(\mathbb{E}^{1,6}) \text{dvol}(\mathbb{E}^4)$$

relates this action to the chirality of the four-dimensional spinor. Under $\text{Spin}(1, 10) \subset \text{Spin}(1, 6) \times \text{Spin}(4)$, the eleven-dimensional spinor representation S_{11} decomposes as

$$S = [S_7 \otimes S_4^+] \oplus [S_7 \otimes S_4^-]$$

where S_4^\pm are the positive (resp. negative) chirality half-spin representations of $\text{Spin}(4) = \text{SU}(2)_- \times \text{SU}(2)_+$, where the \mp refers to (anti)self-duality. This implies that $\text{SU}(2)_\pm$ acts trivially on S_4^\pm , whereas under $\text{SU}(2)_\mp$, S_4^\pm are both isomorphic to the fundamental representation S_3 of $\text{SU}(2) = \text{Spin}(3)$, which is quaternionic and of complex dimension 2. Therefore the product $S_7 \otimes S_4^\pm$ has a real structure and as before $[S_7 \otimes S_4^\pm]$ is the underlying real representation.

As discussed in Appendix B, the Killing spinors of the Kaluza–Klein monopole are in one-to-one correspondence with spinors in S_{11} whose four-dimensional chirality is positive. In other words, the relevant representation of $\text{Spin}(1, 6) \times \text{SU}(2) \times \text{U}(1)$ is isomorphic to $[S_7 \otimes S_3]$ where the $\text{U}(1)$ factor acts trivially. It is easy to determine the weight decomposition of $[S_7 \otimes S_3]$ under a Cartan subalgebra of $\text{Spin}(6) \times \text{SU}(2)$. The general element of such Cartan subalgebra can be written as

$$\theta_1 R_{12} + \theta_2 R_{34} + \theta_3 R_{56} + \theta_4 R_{78} .$$

The weights relative to the canonical dual basis to the above $\{R_{ij}\}$ are given by

$$\text{weights}([S_7 \otimes S_3]) = \left\{ (\mu_1, \mu_2, \mu_3, \mu_4) \left| \mu_i^2 = 1 \right. \right\} , \quad (\text{A.2})$$

where the signs are uncorrelated, for a total of $16 = \dim_{\mathbb{R}}[S_7 \otimes S_3]$ weights.

APPENDIX B. PARALLEL SPINORS IN THE TAUB–NUT SPACE

In this section we derive the expression for the parallel spinors in the Taub–NUT geometry (M, g_{TN}) and exhibit the action of the isometry group on the parallel spinors.

B.1. The parallel spinors. Our starting point is the Taub–NUT metric (3.8) written as a cohomogeneity-one space under the action of $\text{SU}(2)$. In this form, the natural coframe $\{\Theta_m\}$ is given by

$$\Theta_1 = V^{1/2} r \sigma_1 \quad \Theta_2 = V^{1/2} r \sigma_2 \quad \Theta_3 = V^{-1/2} Q \sigma_3 \quad \Theta_4 = V^{1/2} dr .$$

The connection one-forms ω_{mn} , defined by

$$d\Theta_m + \omega_{mn} \wedge \Theta_n = 0 ,$$

are given by

$$\begin{aligned} \omega_{12} &= \frac{1}{2}(1 - V^{-2} + 2V^{-1})\sigma_3 & \omega_{23} &= \frac{1}{2}(1 - V^{-1})\sigma_1 \\ \omega_{13} &= -\frac{1}{2}(1 - V^{-1})\sigma_2 & \omega_{24} &= \frac{1}{2}(1 + V^{-1})\sigma_2 \\ \omega_{14} &= \frac{1}{2}(1 + V^{-1})\sigma_1 & \omega_{34} &= \frac{1}{2}(1 - V^{-1})^2\sigma_3 . \end{aligned}$$

Notice that the anti-self-dual combinations are very simple:

$$\omega_{14} + \omega_{23} = \sigma_1 , \quad \omega_{24} - \omega_{13} = \sigma_2 \quad \text{and} \quad \omega_{12} + \omega_{34} = \sigma_3 ,$$

which together with the structure equations (3.7) for the σ_i , imply that the anti-self-dual components

$$\begin{aligned} \Omega_{12} + \Omega_{34} &= d(\omega_{12} + \omega_{34}) + (\omega_{14} + \omega_{23}) \wedge (\omega_{13} - \omega_{24}) \\ \Omega_{14} + \Omega_{23} &= d(\omega_{14} + \omega_{23}) + (\omega_{12} + \omega_{34}) \wedge (\omega_{24} - \omega_{13}) \\ \Omega_{24} - \Omega_{13} &= d(\omega_{24} - \omega_{13}) + (\omega_{14} + \omega_{23}) \wedge (\omega_{12} + \omega_{34}) \end{aligned} \quad (\text{B.1})$$

of the curvature two-form vanish, showing that Taub–NUT is indeed hyperkähler.

A spinor ε is parallel if and only if it satisfies

$$\nabla \varepsilon := d\varepsilon + \sum_{m < n} \omega_{mn} \Sigma_{mn} \varepsilon = 0 ,$$

where $\Sigma_{mn} = \frac{1}{2} \Gamma_{mn}$ are the spin generators and Γ_n is a basis for the Clifford algebra adapted to the coframe Θ_n and obeying $\Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2\delta_{mn} \mathbb{1}$.

We notice that there is no dr in the expressions for ω_{mn} , whence $\nabla_r = \partial_r$ and hence parallel spinors do not depend on r . This means that we can compute them for any value of r . It is convenient to compute them in the limit $r \rightarrow \infty$ since the asymptotic geometry is flat. Let $\bar{\omega}_{mn}$ denote the connection one-forms in this limit. Noticing that $V \rightarrow 1$ in this limit, we easily find

$$\begin{aligned}\bar{\omega}_{12} &= \sigma_3 & \bar{\omega}_{23} &= 0 \\ \bar{\omega}_{13} &= 0 & \bar{\omega}_{24} &= \sigma_2 \\ \bar{\omega}_{14} &= \sigma_1 & \bar{\omega}_{34} &= 0.\end{aligned}$$

After a short calculation, the parallel spinors in this limit are given by

$$\varepsilon = \exp(-\psi \Sigma_{12}) \exp(-\theta \Sigma_{14}) \exp(-\varphi \Sigma_{12}) \varepsilon_0, \quad (\text{B.2})$$

where ε_0 is a constant spinor.

We now impose that the spinor ε given above be indeed parallel. For this it is convenient to introduce the tensor $T \in \Omega^1(M) \otimes \mathfrak{so}(TM)$:

$$T_{mn} = \omega_{mn} - \bar{\omega}_{mn},$$

which measures the difference between the connection one-forms of the Taub–NUT geometry and its flat asymptotic limit. It is explicitly given by

$$\begin{aligned}T_{12} &= -\frac{1}{2}(1 - V^{-1})^2 \sigma_3 & T_{23} &= \frac{1}{2}(1 - V^{-1}) \sigma_1 \\ T_{13} &= -\frac{1}{2}(1 - V^{-1}) \sigma_2 & T_{24} &= -\frac{1}{2}(1 - V^{-1}) \sigma_2 \\ T_{14} &= -\frac{1}{2}(1 - V^{-1}) \sigma_1 & T_{34} &= \frac{1}{2}(1 - V^{-1})^2 \sigma_3.\end{aligned}$$

For any spinor ε of the form given in (B.2),

$$\nabla \varepsilon = \sum_{m < n} T_{mn} \Sigma_{mn} \varepsilon,$$

whence it will be a parallel spinor in the Taub–NUT geometry if and only if it is annihilated by

$$\begin{aligned}\sum_{m < n} T_{mn} \Sigma_{mn} &= \frac{1}{2}(1 - V^{-1}) \sigma_1 (\Sigma_{23} - \Sigma_{14}) \\ &\quad - \frac{1}{2}(1 - V^{-1}) \sigma_2 (\Sigma_{13} + \Sigma_{24}) - \frac{1}{2}(1 - V^{-1})^2 \sigma_3 (\Sigma_{12} - \Sigma_{34}).\end{aligned}$$

Since the one-forms σ_i are linearly independent, this is equivalent to ε being annihilated by the self-dual combinations $\Sigma_{23} - \Sigma_{14}$, $\Sigma_{13} + \Sigma_{24}$ and $\Sigma_{12} - \Sigma_{34}$; but these three equations are equivalent to the chirality condition $\Gamma_{1234} \varepsilon = -\varepsilon$. Notice that the chiralities of ε and ε_0 agree, whence this equation is equivalent to $\Gamma_{1234} \varepsilon_0 = -\varepsilon_0$. Noticing that the orientation of Taub–NUT is given by

$$\text{dvol}_{\text{TN}} = \Theta_4 \wedge \Theta_1 \wedge \Theta_2 \wedge \Theta_3,$$

we can write the chirality condition more invariantly as

$$\text{dvol}_{\text{TN}} \varepsilon = \varepsilon. \quad (\text{B.3})$$

B.2. The action of the isometry group. Acting on a Killing spinor ε , the expression for the Lie derivative along a Killing vector ξ becomes an algebraic condition. In the case of a parallel spinor, this simplifies to

$$\mathcal{L}_\xi \varepsilon = \frac{1}{4} d\xi^b \varepsilon$$

where ξ^b is the one-form dual to ξ . Let $\varepsilon = \Psi(\psi, \theta, \varphi) \varepsilon_0$ denote a parallel spinor, where $\Psi(\psi, \theta, \varphi)$ denotes the product of exponentials in equation (B.2). Since the action of a Killing vector ξ preserves the space of parallel spinors, it follows that

$$\mathcal{L}_\xi \varepsilon = \Psi(\psi, \theta, \varphi) M_\xi \varepsilon_0,$$

for some *constant* endomorphism M_ξ of the chiral spinor representation. Since M_ξ is constant, the calculation can be simplified by choosing a convenient point in which the expressions simplify. A straightforward calculation reveals that for ξ_i , $i = 1, 2, 3, 4$ given by (3.10), the endomorphisms M_{ξ_i} are given by

$$M_{\xi_1} = -\Sigma_{13} \quad M_{\xi_2} = -\Sigma_{23} \quad M_{\xi_3} = -\Sigma_{12} \quad M_{\xi_4} = 0 .$$

One can check that they satisfy the $\mathfrak{su}(2) \times \mathfrak{u}(1)$ Lie algebra, as expected; in particular, for $i = 1, 2, 3$,

$$[M_{\xi_i}, M_{\xi_j}] = \varepsilon_{ijk} M_{\xi_k} .$$

In summary, the parallel spinors in the Taub–NUT geometry are given by (B.2), where ε_0 (and hence ε) is subject to the chirality condition (B.3). Furthermore the correspondence is equivariant under the action of $\mathfrak{su}(2) \times \mathfrak{u}(1) \subset \mathfrak{so}(4)$, where the $\mathfrak{u}(1)$ is self-dual (and hence acts trivially on positive-chirality spinors) and $\mathfrak{su}(2)$ is anti-self-dual and hence acts on positive-chirality spinors via the fundamental representation.

APPENDIX C. SPIN STRUCTURES ON QUOTIENTS

Let M be a spin manifold and let Γ be a group acting freely on M via orientation-preserving isometries. Is the quotient M/Γ spin? The answer to this question is affirmative if and only if there is a spin bundle on M to which the action of G lifts. Let $P_{\text{SO}}(M)$ denote the bundle of oriented orthonormal frames. This is a principal $\text{SO}(p, q)$ -bundle, where M has signature (p, q) . Let $\vartheta_0 : \text{Spin}(p, q) \rightarrow \text{SO}(p, q)$ be the spin double-cover. A spin structure on M is a principal $\text{Spin}(p, q)$ -bundle $P_{\text{Spin}}(M)$ and a bundle map $\vartheta : P_{\text{Spin}}(M) \rightarrow P_{\text{SO}}(M)$ which restricts to ϑ_0 on the fibres. Since Γ acts via orientation-preserving isometries, it acts on $P_{\text{SO}}(M)$. The question is whether this action lifts to $P_{\text{Spin}}(M)$ in such a way that ϑ is equivariant. This seems to be a difficult question to settle in general; but it is possible to decide for the groups Γ of interest acting on the Kaluza–Klein monopole $M = \mathbb{R}^{1,6} \times \text{TN}$.

Since M is contractible, the frame bundle is trivial:

$$P_{\text{SO}}(M) \cong \text{SO}(1, 10) \times M ,$$

where the isomorphism is given explicitly by choosing an oriented orthonormal frame e_a . Given any other oriented orthonormal frame e'_a , they are related as follows:

$$e'_a(x) = \sum_b e_b(x) L^b_a(x) ,$$

where $L : M \rightarrow \text{SO}(1, 10)$ is a local Lorentz transformation.

As discussed before, the isometry group of M is $(\text{SO}(1, 6) \ltimes \mathbb{R}^7) \times \text{U}(2)$. Given $g \in G$, we have a homonymous map $g : M \rightarrow M$ sending $x \mapsto g \cdot x$ and its derivative map $g_* : T_x M \rightarrow T_{g \cdot x} M$. If $e_a(x)$ is an orthonormal frame at x , then $g_* \cdot e_a(x)$ is an orthonormal frame at $g \cdot x$. Therefore there exists an element $S(g, x) \in \text{SO}(1, 10)$ such that

$$g_* \cdot e_a(x) = \sum_b e_b(g \cdot x) S^b_a(g, x) .$$

In other words, S defines a map $G \times M \rightarrow \text{SO}(1, 10)$. By considering the action of g_* on another frame $e'_a(x)$, we see that the action of G on $P_{\text{SO}}(M) \cong \text{SO}(1, 10) \times M$ consists of left multiplication on $\text{SO}(1, 10)$ and g on M :

$$g_* \cdot (L(x), x) = (S(g, x)L(x), g \cdot x) .$$

Since the map $g \mapsto g_*$ is a homomorphism, the map $S : G \times M \rightarrow \mathrm{SO}(1, 10)$ satisfies the following identity² for all $g_1, g_2 \in G$ and $x \in M$:

$$S(g_1 g_2, x) = S(g_1, g_2 \cdot x) S(g_2, x) . \quad (\text{C.1})$$

Now let $\Gamma \subset G$ be a subgroup and consider the restriction to Γ of the map $S : \Gamma \times M \rightarrow \mathrm{Spin}(1, 10)$. We are interested primarily in connected one-parameter subgroups Γ , which are thus diffeomorphic to either \mathbb{R} or S^1 . Since M is simply-connected, there is a unique spin bundle $P_{\mathrm{Spin}}(M)$. The action of Γ on $P_{\mathrm{SO}}(M)$ will lift to $P_{\mathrm{Spin}}(M)$ if and only if there exists a lift

$$\begin{array}{ccc} & & \mathrm{Spin}(1, 10) \\ & \nearrow ? & \downarrow \vartheta_0 \\ \Gamma \times M & \xrightarrow{S} & \mathrm{SO}(1, 10) \end{array}$$

making the above diagram commutative. If $\Gamma \cong \mathbb{R}$ then the lift exists on purely topological grounds, because both $\Gamma \times M$ and $\mathrm{Spin}(1, 10)$ are simply connected. Therefore we need only consider the cases where $\Gamma \cong S^1$.

Let us trivialise the frame bundle by choosing a frame. Equivalently, we find it more convenient to choose a coframe made out of the dx^μ for $\mathbb{R}^{1,6}$ and the coframe Θ_m introduced in Section B.1 for Taub–NUT. The action of the isometry group on this coframe is given as follows: the translations \mathbb{R}^7 act trivially, whereas $\mathrm{SO}(1, 6)$ acts linearly through the fundamental representation. The action of $\mathrm{U}(2)$ on the Taub–NUT coframe is such that σ_1 and σ_2 are rotated, whereas r and σ_3 are inert. Therefore on this frame only the $\mathrm{U}(1)$ generated by ∂_ψ acts nontrivially and generates a circle action on the plane spanned by $(rV^{\frac{1}{2}}\sigma_1, rV^{\frac{1}{2}}\sigma_2)$. In any case, the map $\Gamma \times M \rightarrow \mathrm{SO}(1, 10)$ is constant on M , and hence equation (C.1) says that it defines a homomorphism $\Gamma \rightarrow \mathrm{SO}(1, 10)$. The question is whether this homomorphism lifts to $\mathrm{Spin}(1, 10)$. This condition is equivalent to the existence of a homomorphism $\Gamma \rightarrow \mathrm{Spin}(1, 10)$ whose image is mapped *isomorphically* under ϑ_0 to the image of Γ in $\mathrm{SO}(1, 10)$.

This question can be analysed as follows. Since Γ is a one-parameter group, its image $\bar{\Gamma} \subset \mathrm{SO}(1, 10)$ is generated by an element X of the Lie algebra $\mathfrak{so}(1, 10)$. We will embed $\mathfrak{so}(1, 10)$ in the Clifford algebra $\mathrm{Cl}(1, 10)$ and we will exponentiate X there: the result is a one-parameter group $\hat{\Gamma} \subset \mathrm{Spin}(1, 10)$ which covers $\bar{\Gamma}$. The question is then whether $\hat{\Gamma}$ is isomorphic to $\bar{\Gamma}$ or covers it twice. It is clear that the latter will be the case if and only if $-1 \in \hat{\Gamma}$.

Let us investigate this question for the two cases of interest:

- (D) $\xi = 2\partial_\psi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$, and
- (E) $\xi = 2\partial_\varphi + p_1 R_{12} + p_2 R_{34} + p_3 R_{56}$.

These Killing vectors define elements of the isometry algebra \mathfrak{g} and hence induce elements in $\mathfrak{so}(1, 10)$ by the derivative at the identity of the map $G \rightarrow \mathrm{SO}(1, 10)$ described above. Introducing the notation γ_{ij} for the generators of the Clifford algebra $\mathrm{Cl}(4)$ corresponding to the Taub–NUT directions, we can write the elements $X \in \mathfrak{so}(1, 10) \subset \mathrm{Cl}(1, 10)$ corresponding to the above Killing vectors as follows

- (D) $X^{(D)} = 2\gamma_{12} + p_1 \Gamma_{12} + p_2 \Gamma_{34} + p_3 \Gamma_{56}$, and
- (E) $X^{(E)} = p_1 \Gamma_{12} + p_2 \Gamma_{34} + p_3 \Gamma_{56}$;

where we have used that ∂_φ acts trivially on the chosen frame. Exponentiating the above elements in the Clifford algebra we obtain subgroups of $\mathrm{Spin}(1, 10)$ with

²This identity says that S is a contravariant functor between the groupoids $G \times M$ and $\mathrm{SO}(1, 10)$. We thank Takashi Kimura for this observation.

elements

$$\begin{aligned} \hat{g}^{(D)}(t) &:= \exp\left(\frac{t}{2}X^{(D)}\right) = (\mathbb{1} \cos t + \gamma_{12} \sin t) \left(\mathbb{1} \cos \frac{p_1 t}{2} + \Gamma_{12} \sin \frac{p_1 t}{2}\right) \\ &\quad \times \left(\mathbb{1} \cos \frac{p_2 t}{2} + \Gamma_{34} \sin \frac{p_2 t}{2}\right) \left(\mathbb{1} \cos \frac{p_3 t}{2} + \Gamma_{56} \sin \frac{p_3 t}{2}\right) , \end{aligned}$$

and

$$\begin{aligned} \hat{g}^{(E)}(t) &:= \exp\left(\frac{t}{2}X^{(E)}\right) = \left(\mathbb{1} \cos \frac{p_1 t}{2} + \Gamma_{12} \sin \frac{p_1 t}{2}\right) \\ &\quad \times \left(\mathbb{1} \cos \frac{p_2 t}{2} + \Gamma_{34} \sin \frac{p_2 t}{2}\right) \left(\mathbb{1} \cos \frac{p_3 t}{2} + \Gamma_{56} \sin \frac{p_3 t}{2}\right) , \end{aligned}$$

for $t \in [0, 4\pi]$. Each of these groups is either a double-cover of the image $\bar{\Gamma}$ of Γ in $\text{SO}(1, 10)$ or isomorphic to it. It will be a double cover if and only if it contains $-\mathbb{1}$, since this is the only other point in the pre-image under ϑ_0 of the identity in $\text{SO}(1, 10)$. A moment's thought reveals that this can only occur for $t = 2\pi$ where

$$g^{(D)}(2\pi) = g^{(E)}(2\pi) = (-1)^{p_1+p_2+p_3} \mathbb{1} .$$

Therefore we conclude that this will contain $-\mathbb{1}$ if and only if the sum of the p_i is odd. In other words, M/Γ is spin if and only if $p_1 + p_2 + p_3$ is even.

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